The Value of a Bond with Default Probability

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Let X represent the present value of a bond's cash flow stream. When you have a default probability then X becomes a random variable with a range or (as a simplifying assumption) a finite number of possible values. The way to value the bond in this case is to take each possible value of X, multiply it by its probability and sum the results. In other words the value of the bond should equal the mathematical expectation of X.

To illustrate the idea, consider the case of a bond with 4 coupon payments until maturity. Let p be the probability that the bond survives from one coupon payment to the next and let X_i $(i = 0, 1, 2, 3, 4)$ be the value of X given that the bond defaults after making its i^{th} coupon payment. The possible paths this bond can take is illustrated in figure 1.

The expectation of X can then be expressed as:

$$E[X] = X_0(1-p) + X_1 p(1-p) + X_2 p^2(1-p) + X_3 p^3(1-p) + X_4 p^4 \quad (1)$$

This formula can also be written as

$$E[X] = X_0 + (X_1 - X_0)p + (X_2 - X_1)p^2 + (X_3 - X_2)p^3 + (X_4 - X_3)p^4 \quad (2)$$

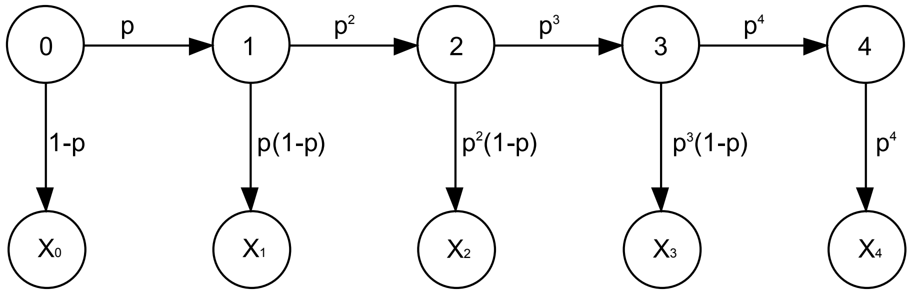

Figure 1: Possible paths of 4 coupon payment bond with constant probability of surviving between payments

To come up with expressions for the X_i, the following variables need to be defined:

$$
\begin{aligned}
F &= \text{face value of the bond} \\
C &= \text{coupon payment} \\
I &= C/F = \text{coupon interest} \\
R &= \text{recovery rate} = [0,1] \\
f &= \text{risk free interest rate} \\
d &= 1/(1+f) = \text{discount factor}
\end{aligned}
$$

I define the recovery rate as the fraction of the face value that is payed if the bond defaults so it must be in the range of 0 to 1. Assuming that R remains constant, the values of X_i are

$$
\begin{aligned}
X_0 &= RF \\
X_1 &= (C + RF)/d \\
X_2 &= C/d + (C + RF)/d^2 \\
X_3 &= C/d + C/d^2 + (C + RF)/d^3 \\
X_4 &= C/d + C/d^2 + C/d^3 + (C + F)/d^4
\end{aligned}
$$

If you put these into the formula for $E[X]$, you get

$$
\begin{aligned}
E[X] &= C\left((p/d) + (p/d)^2 + (p/d)^3 + (p/d)^4\right) + && (3)\\
&\quad RF(1-p)\left(1 + (p/d) + (p/d)^2 + (p/d)^3\right) + && (4)\\
&\quad F(p/d)^4 && (5)
\end{aligned}
$$

Now it should be obvious how to extend this to the case of N coupons instead of just 4. The formula in more compact form is

$$
E[X] = C\sum_{k=1}^{N}(p/d)^k + RF(1-p)\sum_{k=0}^{N-1}(p/d)^k + F(p/d)^N \tag{6}
$$

Each of the sums in this formula is a geometric series that can be collapsed into a single term. The formula for collapsing a general geometric series is

$$
\sum_{k=0}^{N} a^k = 1 + a + a^2 + \ldots + a^N = \frac{1-a^{N+1}}{1-a} \tag{7}
$$

Using this result in the formula for $E[X]$ gives you

$$
E[X] = C(p/d)\frac{1-(p/d)^N}{1-(p/d)} + RF(1-p)\frac{1-(p/d)^N}{1-(p/d)} + F(p/d)^N \tag{8}
$$

This expression can be simplified a little by dividing through by F and making the substitutions $z = p/d$, $I = C/F$. With a little algebra, you get

$$
\frac{E[X]}{F} = (Iz + R(1-dz))\,\frac{1-z^N}{1-z} + z^N \tag{9}
$$

With $E[X]$ equal to the price of the bond, this equation can be solved numerically for z which then gives the survival probability as $p = dz$. Given an assumption of what the survival probability is, the equation can be used to calculate $E[X]$ which is the fair price for the bond.

To get the default probability for the bond, simply subtract the survival probability from 1, default probability $= 1 - p$. The cumulative default probability, or the probability that the bond defaults anytime within the next n coupon periods is $1 - p^n$.

There are several ways to test the formula for logical consistency. First look at the case where the survival probability is zero so that with $z = 0$ the formula reduces to

$$\frac{E[X]}{F} = R \tag{10}$$

This is logical since when default is immanent the price should just equal the recovery amount.

In the case where survival is certain and the risk free rate is zero you have $z = 1$ and

$$\frac{E[X]}{F} = NI + 1 \tag{11}$$

The price here is equal to the total of the coupon payments plus the face value, as you would expect.